Orbital Calculations Notes

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1 Period of Celestial Objects

In order to calculate the orbital periods of celestial objects, assuming a two-object problem, we can use the *Kepler's Third Law*.

$$\frac{P^2}{a^3} = \frac{4\pi^2}{G(M+m)} \tag{1}$$

where a is the *semi-axis major* of the orbital ellipse, G is the *gravitation constant* and M and m are the bigger and smaller mass of the two objects.

Given this formula, we can compute the period with

$$P = 2\pi\sqrt{\frac{a^3}{G(M+m)}} \tag{2}$$

If, instead of a we know the *periastra* of the orbit r_{min} (the closest point between the two objects), we can compute the period as

$$P = 2\pi\sqrt{\frac{r_{min}^3}{(1-\epsilon)^3 G(M+m)}} \tag{3}$$

where ϵ is the ellipse eccentricity.

2 Position Over Time

Using the *Kepler Law of Motion* we can compute the object location in function of time.

We need to follow the following steps:

2.1 Compute the Median Anomaly

We compute the *median anomaly* M with the following formula

$$M(t) = \frac{2\pi}{P}t \tag{4}$$

2.2 Compute the Eccentric Anomaly

We now compute the *Eccentric Anomaly* E solving the following equation:

$$M(t) = E(t) - \epsilon \sin(E(t)) \tag{5}$$

There is no closed form for this. We can solve this through numerical algorithms or using the following approximation (obtained by iteration).

$$E = M + \epsilon \sin(M) + \epsilon^2 \sin(M)\cos(M) + \frac{1}{2}\epsilon^3 \sin(M)\left(3\cos^2(M) - 1\right) \tag{6}$$

This formula can be optimized for computation:

$$E = M + \epsilon \sin(M)\left(1 - \epsilon\left(\frac{\epsilon}{2} + \cos(M)\left(1 + \frac{3\epsilon}{2}\cos(M)\right)\right)\right) \tag{7}$$

2.3 Compute the True Anomaly

We can compute the angle in astrocentric polar coordinates (*True Anomaly*) by solving:

$$(1-\epsilon)\tan^2\left(\frac{\theta}{2}\right) = (1+\epsilon)\tan^2\left(\frac{E}{2}\right) \tag{8}$$

Solved by θ:

$$\tan\left(\frac{\theta}{2}\right) = \sqrt{\frac{1+\epsilon}{1-\epsilon}}\tan\left(\frac{E}{2}\right) \tag{9}$$

$$\theta = 2\ \arg\left(\sqrt{1-e}\ \cos\frac{E}{2}, \sqrt{1+e}\sin\frac{E}{2}\right) \tag{10}$$

where $\arg(x, y)$ is the polar argument of the vector (x, y), also known as `atan2` in many programming languages.

2.4 Compute the Distance

Then we can compute the distance from the orbit center:

$$r = a(1 - \epsilon\cos(E)) \tag{11}$$

Periastra is when $E = 0$ and then $r_{min} = a(1-\epsilon)$. Apoastra is when $E = \pi$ and then $r_{max} = a(1+\epsilon)$.

3 Lunar Phases

If the orbit is of a moon-like object, we may be interested in computing lunar phases.

3.1 Simplified Model

In this model we avoid taking into account the revolution of the main planet, therefore phases are only dependent on the angle between Sun and main body. We can then compute the amount of in shadow surface of the moon seen from the main body.

$$\%pn = \frac{|\theta|}{\pi} \tag{12}$$